On some Rajchman measures and equivalent Salem's problem

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Abstract

We construct certain Rajchman measures by using integrability properties of the Fourier and Fourier-Stieltjes transforms. In particular, we state a problem and prove that it is equivalent to the known and still unsolved question posed by R. Salem (*Trans. Amer. Math. Soc.* **53** (3) (1943), p. 439) whether Fourier-Stieltjes coefficients of the Minkowski's question mark function vanish at infinity.

Keywords: Rajchman measure, Minkowski question mark function, Salem's problem, Fourier-Stieltjes transform, modified Bessel function, index integral, Fourier-Stieltjes coefficients

Mathematics subject classification: 33C10, 42A16, 42B10, 44A15

1 Introduction and auxiliary results

It is well known in the elementary theory of the Fourier-Stieltjes integrals that if $h(x)$ is absolutely continuous then

$$g(t) = \int_{\Omega} e^{ixt} dh(x), \ \Omega \subset \mathbb{R}, \ t \in \mathbb{R} \tag{1}$$

tends to zero as $|t| \to \infty$, because in this case the Fourier-Stieltjes transform $g(t)$ is an ordinary Fourier transform of an integrable function. Thus $h(x)$ supports a measure whose Fourier transform vanishes at infinity. Such measures are called Rajchman measures (see details, for instance, in [3]). However, when h is continuous, the situation is quite different and the classical Riemann-Lebesgue lemma for the class L_1, in general, cannot be applied. The question is quite delicate when it concerns singular monotone functions (see [8], Ch. IV). For such singular measures there are various examples and the Fourier-Stieltjes transform need not

tend to zero, although there do exist measures for which it goes to zero. For instance, Salem [6] [7] gave examples of singular functions, which are strictly increasing and whose Fourier coefficients still do not vanish at infinity. On the other hand, Menchoff in 1916 [4] gave a first example of a singular distribution whose coefficients vanish at infinity. Wiener and Wintner (see also [2]) proved in 1938 that for every $\varepsilon > 0$ there exists a singular monotone function such that its Fourier coefficients behave as $n^{-\frac{1}{2}+\varepsilon},\ n \to \infty$.

Our goal in this paper is to construct some Raijchman's measures, which are associated with continuous functions of bounded variation. In particular, we will prove that the famous Minkowski' s question mark function $?(x)$ [1] is a Raijchman measure if and only if its Forier-Stieltjes transform has a limit at infinity, and then, of course, the limit should be zero. This probably can give an affirmative answer on the question posed by Salem in 1943 [6].

The Minkowski question mark function $?(x) : [0,1] \mapsto [0,1]$ is defined by [1]

$$?([0, a_1, a_2, a_3, \ldots]) = 2\sum_{i=1}^{\infty} (-1)^{i+1} 2^{-\sum_{j=1}^{i} a_j}, \tag{2}$$

where $x = [0, a_1, a_2, a_3, \ldots]$ stands for the representation of x by a regular continued fraction. We will keep the notation $?(x)$, which was used in the original Salem's paper [6], mildly resisting the temptation of changing it and despite this symbol is quite odd to denote a function in such a way. It is well known that $?(x)$ is continuous, strictly increasing and singular with respect to Lebesgue measure. It can be extended on $[0, \infty]$ by using the following functional equations

$$?(x) = 1 - ?(1-x), \tag{3}$$

$$?(x) = 2?\left(\frac{x}{x+1}\right), \tag{4}$$

$$?(x) + ?\left(\frac{1}{x}\right) = 2, \ x > 0. \tag{5}$$

When $x \to 0$, it decreases exponentially $?(x) = O\left(2^{-1/x}\right)$. Key values are $?(0) = 0,\ ?(1) = 1,\ ?(\infty) = 2$. For instance, from (3) and asymptotic behavior of the Minkowski function $?(x)$ near zero one can easily get the finiteness of the following integrals

$$\int_0^1 x^{\lambda}\, d?(x) < \infty, \ \lambda \in \mathbb{R}, \tag{6}$$

$$\int_0^1 (1-x)^{\lambda}\, d?(x) < \infty, \ \lambda \in \mathbb{R}. \tag{7}$$

Further, as was proved by Salem [6], the Minkowski question mark function satisfies the Hölder condition

$$|?(x) - ?(y)| < C|x-y|^{\alpha}, \tag{8}$$

of order

$$\alpha = \frac{\log 2}{2\log\frac{\sqrt{5}+1}{2}}, \tag{9}$$

where $C > 0$ is an absolute constant. We will deal in the sequel with the following Fourier -Stieltjes transforms of the Minkowski question mark function

$$f(t) = \int_0^1 e^{ixt}\ d?(x), \quad F(t) = \int_0^\infty e^{ixt}\ d?(x), \quad t \in \mathbb{R}, \tag{10}$$

$$f_c(t) = \int_0^1 \cos xt\ d?(x), \quad F_c(t) = \int_0^\infty \cos xt\ d?(x),\ t \in \mathbb{R}_+, \tag{11}$$

$$f_s(t) = \int_0^1 \sin xt\ d?(x), \quad F_s(t) = \int_0^\infty \sin xt\, \mathrm{d}?(x),\ t \in \mathbb{R}_+, \tag{12}$$

where all integrals converge absolutely and uniformly with respect to t because of straightforward estimates

$$|f(t)| \le \int_0^1 d?(x) = 1, \quad |F(t)| \le \int_0^\infty d?(x) = 2,$$

$$|f_c(t)| \le 1, \quad |F_c(t)| \le 2,$$

$$|f_s(t)| \le 1, \quad |F_s(t)| \le 2.$$

Further we observe that functional equation (3) easily implies $f(t) = e^{it}f(-t)$ and therefore $e^{-it/2}f(t) \in \mathbb{R}$. So, taking the imaginary part we obtain the equality

$$\cos\left(\frac{t}{2}\right) f_s(t) = \sin\left(\frac{t}{2}\right) f_c(t). \tag{13}$$

Hence, for instance, letting $t = 2\pi n,\ n \in \mathbb{N}_0$ it gives $f_s(2\pi n) = 0$ and $f_c(2\pi n) = d_n$. In 1943 Salem asked [6] whether $d_n \to 0$, as $n \to \infty$.

Further, by using functional equations (4), (5) for the Minkowski function we derive the following useful relations

$$\int_0^1 e^{ixt}\ d?(x) = \int_0^\infty e^{ixt}\ d?(x) - \int_1^\infty e^{ixt}\ d?(x)$$

$$= \int_0^\infty e^{ixt}\ d?(x) + e^{it}\int_0^\infty e^{ixt}\ d?\left(\frac{1}{x+1}\right)$$

$$= \int_0^\infty e^{ixt}\ d?(x) + \frac{e^{it}}{2}\int_0^\infty e^{ixt}\ d?\left(\frac{1}{x}\right)$$

$$= \left(1-\frac{e^{it}}{2}\right)\int_0^\infty e^{ixt}\ d?(x),$$

which imply the functional equation

$$F(t) = \frac{2f(t)}{2-e^{it}}. \tag{14}$$

Taking real and imaginary parts in (14) and employing functional equation (3) it is not difficult to deduce the following important equalities for the Fourier-Stieltjes transforms (11), (12)

$$F_c(t) = \frac{2}{5-4\cos t}\ f_c(t), \tag{15}$$

$$F_s(t) = \frac{6}{5-4\cos t}\ f_s(t). \tag{16}$$

Indeed, we have, for instance

$$\int_0^\infty \cos xt\ d?(x) = \frac{2}{5-4\cos t}\left[(2-\cos t)\int_0^1 \cos xt\ d?(x) - \sin t\int_0^1 \sin xt\ d?(x)\right]$$

$$= \frac{2}{5-4\cos t}\left[2\int_0^1 \cos xt\ d?(x) - \int_0^1 \cos t(1-x)\ d?(x)\right]$$

$$= \frac{2}{5-4\cos t}\int_0^1 \cos xt\ d?(x)$$

and this yields relation (15). Analogously we get (16). In particular, letting $t = 2\pi n, n \in \mathbb{N}_0$ in (15), (16) we find accordingly

$$\int_1^\infty \cos(2\pi nx)\ d?(x) = \int_0^1 \cos(2\pi nx)\ d?(x),$$

$$\int_1^\infty \sin(2\pi n x)\ d?(x) = 5 \int_0^1 \sin(2\pi n x)\ d?(x) = 0$$

via (13). Generally, equalities (15), (16) yield

$$\int_1^\infty \cos xt\ d?(x) = \frac{1-8\sin^2(t/2)}{1+8\sin^2(t/2)} \int_0^1 \cos xt\ d?(x),$$

$$\int_1^\infty \sin xt\ d?(x) = \frac{5-8\sin^2(t/2)}{1+8\sin^2(t/2)} \int_0^1 \sin xt\ d?(x).$$

respectively. For instance,

$$\int_1^\infty \cos(xt_m)\ d?(x) = 0,$$

$$\int_1^\infty \sin(xt_k)\ d?(x) = 0$$

for any $t_m,\ t_k$, which are roots of the corresponding equations

$$\sin(t_m/2) = \pm 1/(2\sqrt{2}),\ \sin(t_k/2) = \pm\sqrt{5/8}, \quad m, k \in \mathbb{N}.$$

Further, since (see (14), (15), (16))

$$\frac{1}{2}\ |F(t)| \le |f(t)| \le \frac{3}{2}|F(t)|, \tag{17}$$

$$\frac{1}{2}\ |F_c(t)| \le |f_c(t)| \le \frac{9}{2}|F_c(t)|, \tag{18}$$

$$\frac{1}{6}\ |F_s(t)| \le |f_s(t)| \le \frac{3}{2}|F_s(t)|, \tag{19}$$

then Fourier-Stieltjes transforms of the Minkowski question mark function over $(0,1)$ tend to zero when $|t| \to \infty$ if and only if the same property is guaranteed by Fourier-Stieltjes transforms over $(0,\infty)$.

We will show in the next section that the finite Fourier-Stieltjes transform can be treated with the use of the so-called Lebedev-Stieltjes integrals, involving the modified Bessel function

$K_{i\tau}(x)$ of the pure imaginary index [11]. It is known that the modified Bessel function $K_\mu(z)$ satisfies the differential equation

$$z^2\frac{d^2u}{dz^2}+z\frac{du}{dz}-(z^2+\mu^2)u=0$$

and has the following asymptotic behavior

$$K_\mu(z)=\left(\frac{\pi}{2z}\right)^{1/2}e^{-z}[1+O(1/z)],\qquad z\to\infty, \tag{20}$$

$$K_\mu(z)=O(z^{-|\mathrm{Re}\mu|}),\ z\to 0,\ \mu\neq 0, \tag{21}$$

$$K_0(z)=-\log z+O(1),\ z\to 0. \tag{22}$$

When $|\tau|\to\infty$ and $x>0$ is fixed it behaves as

$$K_{i\tau}(x)=O\left(\frac{e^{-\pi|\tau|/2}}{\sqrt{|\tau|}}\right). \tag{23}$$

We will appeal in the sequel to the uniform inequality for the modified Bessel function

$$|K_{i\tau}(x)|\le\frac{x^{-1/4}}{\sqrt{\sinh\pi\tau}},\ x,\tau>0 \tag{24}$$

and its representation via the following Fourier cosine integral

$$\cosh\left(\frac{\pi\tau}{2}\right)K_{i\tau}(x)=\int_0^\infty\cos\tau u\cos(x\sinh u)du,\ x>0. \tag{25}$$

Furthermore, employing relation (2.16.48.20) in [5] and making differentiation by a parameter we derive useful integral with respect to an index of the modified Bessel function

$$\begin{aligned}&\frac{1}{\pi}\int_{-\infty}^{\infty}\tau e^{\lambda\tau}\ \left(t+(1+t^2)^{1/2}\right)^{i\tau}K_{i\tau}(x)d\tau\\&\quad=x\exp\left(-x\left[(1+t^2)^{1/2}\cos\lambda-it\sin\lambda\right]\right)\left[(1+t^2)^{1/2}\sin\lambda+it\cos\lambda\right],\quad x,t>0\end{aligned} \tag{26}$$

where $0\le\lambda<\frac{\pi}{2}$.

2 Some Rajchman measures

In this section we prove several theorems, characterizing Rajchman measures, which are associated with Fourier-Stieltjes integrals over finite and infinite intervals.

We begin with the following general result.

Theorem 1. *Let φ be a real- valued continuous integrable function of bounded variation on $(0,\infty)$ vanishing at infinity. Then φ supports a Rajchman measure relatively its Fourier-Stieltjes transform*

$$\Phi(t) = \int_0^\infty e^{ixt}\ d\varphi(x), \tag{27}$$

if and only if it has a limit at infinity ($|t| \to \infty$).

Proof. Without loss of generality we prove the theorem for positive t. Evidently, the necessity is trivial and we will prove the sufficiency. Suppose that the limit of $\Phi(t)$ when $t \to +\infty$ exists. Since $\Phi(t) = \Phi_c(t) + i\Phi_s(t)$, where

$$\Phi_c(t) = \int_0^\infty \cos xt\ d\varphi(x), \tag{28}$$

$$\Phi_s(t) = \int_0^\infty \sin xt\ d\varphi(x), \tag{29}$$

we will treat these transforms separately. Taking (28) and integrating by parts we get

$$\Phi_c(t) = -\varphi(0) + t\int_0^\infty \varphi(x)\sin xt\ dx. \tag{30}$$

However, since $\varphi \in L_1(\mathbb{R}_+)$, we appeal to the integrated form of the Fourier formula (cf. [9], Th. 22) to write for all $x \geq 0$

$$\int_0^x \varphi(y)\ dy = \frac{2}{\pi}\int_0^\infty \frac{1-\cos yx}{y}\int_0^\infty \varphi(u)\sin uy\ du.$$

But taking into account the previous equality after simple change of variable we come out with the relation

$$\frac{1}{x}\int_0^x \varphi(y)\ dy = \frac{2}{\pi}\int_0^\infty \frac{1-\cos y}{y^2}\left[\varphi(0) + \Phi_c\left(\frac{y}{x}\right)\right] dy,\ x > 0.$$

Minding the value of elementary Feijer type integral

$$\frac{2}{\pi}\int_0^\infty \frac{1-\cos y}{y^2} dy = 1,$$

we establish an important equality

$$\frac{1}{x}\int_0^x [\varphi(y) - \varphi(0)]\ dy = \frac{2}{\pi}\int_0^\infty \Phi_c\left(\frac{y}{x}\right)\frac{1-\cos y}{y^2} dy,\ x > 0. \tag{31}$$

Meanwhile, the left-hand side of (31) is evidently goes to zero when $x \to 0+$ via the continuity of φ on $[0,\infty)$. Further, since φ is of bounded variation on $(0,\infty)$ we obtain the uniform estimate

$$|\Phi_c(t)| \leq \int_0^\infty\ dV_\varphi(x) = \Phi_0,$$

where $V_\varphi(x)$ is a variation of φ on $[0,x]$ and $\Phi_0 > 0$ is a total variation of φ. This means that $\Phi_c(t)$ is continuous and bounded on $\mathbb{R}_+$. Furthermore, the integral with respect to x in the right-hand side of (31) converges absolutely and uniformly by virtue of the Weierstrass test. Consequently, since $\Phi_c(t)$ has a limit at infinity, which is finite, say a, one can pass to the limit through equality (31) when $x \to 0+$. Hence we find

$$\lim_{x\to 0+} \frac{1}{x}\int_0^x [\varphi(y)-\varphi(0)]\, dy = \frac{2a}{\pi}\int_0^\infty \frac{1-\cos y}{y^2} dy = a = 0.$$

In order to complete the proof, we need to verify whether the Fourier sine transform (29) tends to zero as well. To do this, we appeal to the corresponding integrated form of the Fourier formula for the Fourier cosine transform

$$-\int_0^x \varphi(y)\, dy = \frac{2}{\pi}\int_0^\infty \frac{\sin yx}{y^2}\Phi_s(y)\, dy,\ x>0, \tag{32}$$

where after integration by parts $\Phi_s(t)$ turns to be represented as follows

$$\Phi_s(t) = -t\int_0^\infty \varphi(u)\cos ut\, du,\ t>0. \tag{33}$$

Hence it is easily seen that $\Phi_s(t) = O(t),\ t \to 0+$ and since $|\Phi_s(t)| \le \Phi_0$ we have that $\frac{\Phi_s(t)}{t} \in L_2(\mathbb{R}_+)$. This means that the integral in the right-hand side of (32) converges absolutely and uniformly by $x \ge 0$. After simple change of variable we split the integral in the right-hand side of (32) on two integrals to obtain

$$-\frac{1}{x}\int_0^x \varphi(y)\, dy = \frac{2}{\pi}\int_0^1 \frac{\sin y}{y^2}\Phi_s\left(\frac{y}{x}\right)\, dy + \frac{2}{\pi}\int_1^\infty \frac{\sin y}{y^2}\Phi_s\left(\frac{y}{x}\right)\, dy.$$

Considering again $x>0$ sufficiently small and splitting the integral over $(0,1)$ on two more integrals over $(0, x\log^\gamma(1/x))$ and $(x\log^\gamma(1/x), 1)$, where $0<\gamma<1$, we derive the equality

$$\frac{2}{\pi}\int_{x\log^\gamma(1/x)}^1 \frac{\sin y}{y^2}\Phi_s\left(\frac{y}{x}\right)\, dy = -\frac{1}{x}\int_0^x \varphi(y)\, dy - \frac{2}{\pi}\int_1^\infty \frac{\sin y}{y^2}\Phi_s\left(\frac{y}{x}\right)\, dy$$

$$-\frac{2}{\pi}\int_0^{x\log^\gamma(1/x)} \frac{\sin y}{y^2}\Phi_s\left(\frac{y}{x}\right)\, dy.$$

Minding the inequality (see (33)) $|\Phi_s(t)| \le t||\varphi||_{L_1(\mathbb{R}_+)},\ t\ge 0$, the right-hand side of the latter equality has the straightforward estimate

$$\left|\frac{1}{x}\int_0^x \varphi(y)\, dy + \frac{2}{\pi}\int_1^\infty \frac{\sin y}{y^2}\Phi_s\left(\frac{y}{x}\right)\, dy \right.$$
$$\left. + \frac{2}{\pi}\int_0^{x\log^\gamma(1/x)} \frac{\sin y}{y^2}\Phi_s\left(\frac{y}{x}\right)\, dy\right| \le \sup_{y\ge 0}|\varphi(y)| + \frac{2}{\pi}\left[\Phi_0 + ||\varphi||_{L_1(\mathbb{R}_+)}\log^\gamma(1/x)\right]. \tag{34}$$

On the other hand, via the first mean value theorem

$$\frac{2}{\pi}\left|\int_{x\log^\gamma(1/x)}^1 \frac{\sin y}{y^2}\Phi_s\left(\frac{y}{x}\right)\ dy\right| = \frac{2}{\pi}|\Phi_s(\xi(x))|\int_{\log^\gamma(1/x)}^1 \frac{\sin y}{y^2}\ dy,$$

where

$$\log^\gamma\left(\frac{1}{x}\right) \leq \xi(x) \leq \frac{1}{x}.$$

Meanwhile, we have

$$\frac{2}{\pi}\int_{x\log^\gamma(1/x)}^1 \frac{\sin y}{y^2}\ dy > \frac{2\sin 1}{\pi}\int_{\log^\gamma(1/x)}^1 \frac{dy}{y} = \frac{2\sin 1}{\pi}\log\left(\frac{1}{x\log^\gamma(1/x)}\right).$$

Consequently, combining with (34) we find

$$|\Phi_s(\xi(x))| < \frac{1}{\sin 1}\left[\frac{\pi}{2}\sup_{y\geq 0}|\varphi(y)| + \Phi_0 + ||\varphi||_{L_1(\mathbb{R}_+)}\log^\gamma(1/x)\right]\log^{-1}\left(\frac{1}{x\log^\gamma(1/x)}\right)$$
$$= o(1),\ x\to 0+. \tag{35}$$

Thus making $x\to 0+$ we get $\xi(x)\to+\infty$ and therefore there is a subsequence $t_n=\xi(x_n)\to\infty$ such that $\lim_{n\to+\infty}|\Phi_s(t_n)|=0$. But since the limit of $\Phi_s(t)$ exists, when $t\to+\infty$ it will be zero. So φ supports a Raijcman measure and the theorem is proved. □

Corollary 1. *Under conditions of Theorem 1 φ supports a Rajchman measure if and only if two limits*

$$\lim_{t\to+\infty} t\int_0^\infty \varphi(x)\sin xt\ dx,\qquad \lim_{t\to+\infty} t\int_0^\infty \varphi(x)\cos xt\ dx$$

exist simultaneously (if so, they equal to $\varphi(0)$ and 0, respectively).

More general result deals with the smoothness of the Fourier-Stieltjes transform and a behavior at infinity of its derivatives.

We have

Corollary 2. *Let $n\in\mathbb{N}_0,\ \varphi(x),\ x\geq 0$ be a real-valued continuous function such that $x^m\varphi(x)$ is of bounded variation on $[0,\infty)$ for each $m=0,1,\ldots,n$. If $\varphi(x)=o(x^{-n}), x\to\infty$ and $x^n\varphi(x)\in L_1(\mathbb{R}_+)$, then the corresponding Fourier-Stieltjes transform (27) $\Phi(t)$ is n times differentiable on $\mathbb{R}_+$, its n-th order derivative is equal to*

$$\Phi^{(n)}(t) = \int_0^\infty (ix)^n e^{itx}\ d\varphi(x) \tag{36}$$

and vanishes at infinity if and only if there exists a limit of the integral

$$\Psi_n(t) = \int_0^\infty e^{itx}\ d\left(x^n\varphi(x)\right)$$

when $|t|\to\infty$.

Proof. In fact, under conditions of the corollary one can differentiate n times under the integral sign in the Fourier-Stieltjes transform (27) via the absolute and uniform convergence. Precisely, this circumstance is guaranteed by the estimate

$$\left|\int_0^\infty (ix)^m e^{itx}\ d\varphi(x)\right| = \left|\int_0^\infty e^{itx}\ d\left((ix)^m\varphi(x)\right)\right.$$

$$\left.-m\ i^m \int_0^\infty x^{m-1}\varphi(x)e^{itx}\ dx\right| \le Var_{[0,\infty)}\left(x^m\varphi(x)\right)$$

$$+m\int_0^\infty x^{m-1}|\varphi(x)|\ dx = \Phi_m < \infty,\ m = 0, 1, \ldots, n,$$

where the latter integral is finite since $x^n\psi(x) \in L_1(\mathbb{R}_+)$ and φ is continuous. Thus (36) holds and in order to complete the proof we write it as

$$\Phi^{(n)}(t) = i^n \left[\int_0^\infty e^{itx}\ d\left(x^n\varphi(x)\right) - n\int_0^\infty x^{n-1}\varphi(x)e^{itx}dx\right].$$

The second integral of this equality tends to zero when $t \to \infty$ via the Riemann-Lebesgue lemma. Therefore, $\Phi^{(n)}(t) = o(1),\ t \to \infty,\ n \in \mathbb{N}_0$ if and only if the first integral has a limit at infinity and this limit is certainly zero. □

To finish a characterization of such a kind of Rajchman measures we will prove one more theorem involving general finite Fourier and Fourier-Stieltjes transforms of a continuous function of bounded variation on $[0, 1]$

$$\hat{\psi}(t) = \int_0^1 e^{itx}\psi(x)dx, \tag{37}$$

$$\Psi(t) = \int_0^1 e^{itx}\ d\psi(x). \tag{38}$$

Theorem 2. *Let $\psi(x)$ be a continuous function of bounded variation on $[0, 1]$ such that $\psi(0) = \psi(1) = 0$ and $\psi(x)/x \in L_2(0, 1)$. If a derivative of Fourier transform (37) of ψ satisfies conditions:*

$$i)\quad t^m\hat{\psi}^{(m)}(t) \in L_1(1,\infty),\ m = 0, 1, 2, \quad ii)\quad t^m\hat{\psi}^{(m-1)}(t) = o(1),\ t \to \infty,\ m = 1, 2, \tag{39}$$

then ψ is a Rajchman measure, i.e. $\Psi(t) = o(1),\ |t| \to \infty.$

Proof. Without loss of generality we prove the theorem for positive t. Taking (38) we integrate by parts and eliminating integrated terms come out with the equality

$$\Psi(t) = -it\int_0^1 e^{itx}\psi(x)dx.$$

Meanwhile, passing to the limit through equality (26) when $\lambda \to \frac{\pi}{2}-$, we find

$$\begin{aligned}\frac{1}{\pi} \lim_{\lambda \to \frac{\pi}{2}-} \int_{-\infty}^{\infty} \tau e^{\lambda\tau} \left(t + (1+t^2)^{1/2}\right)^{i\tau} K_{i\tau}(x) d\tau \\ = x(1+t^2)^{1/2}\, e^{ixt}, \quad x, t > 0.\end{aligned} \tag{40}$$

Hence

$$\begin{aligned}\Psi(t) = \frac{t}{\pi i (1+t^2)^{1/2}} \int_0^1 \psi(x) \\ \times \lim_{\lambda \to \frac{\pi}{2}-} \int_{-\infty}^{\infty} \tau e^{\lambda\tau} \left(t + (1+t^2)^{1/2}\right)^{i\tau} K_{i\tau}(x) d\tau\, \frac{dx}{x}.\end{aligned} \tag{41}$$

But since for each $x, t > 0$ and $0 \le \lambda < \frac{\pi}{2}$ (see (26))

$$\left| \int_{-\infty}^{\infty} \tau e^{\lambda\tau} \left(t + (1+t^2)^{1/2}\right)^{i\tau} K_{i\tau}(x) d\tau \right| \le x \left[t + (1+t^2)^{1/2}\right]$$

and ψ is integrable we can take out the limit in (41) having the representation

$$\Psi(t) = \frac{t}{\pi i (1+t^2)^{1/2}} \lim_{\lambda \to \frac{\pi}{2}-} \int_0^1 \psi(x) \int_{-\infty}^{\infty} \tau e^{\lambda\tau} \left(t + (1+t^2)^{1/2}\right)^{i\tau} K_{i\tau}(x) d\tau\, \frac{dx}{x}. \tag{42}$$

A change of the order of integration in (42) is allowed via Fubini's theorem and can be easily justified employing inequality (24) and integrability of the function $\psi(x)x^{-5/4}$ over $(0, 1)$, which is guaranteed by the condition $\psi(x)/x \in L_2(0, 1)$. Consequently,

$$\begin{aligned}\Psi(t) = \frac{t}{\pi i (1+t^2)^{1/2}} \lim_{\lambda \to \frac{\pi}{2}-} \int_{-\infty}^{\infty} \tau e^{\lambda\tau} \left(t + (1+t^2)^{1/2}\right)^{i\tau} \\ \times \int_0^1 K_{i\tau}(x)\psi(x)\, \frac{dx}{x}\, d\tau.\end{aligned} \tag{43}$$

However, the inner integral with respect to x in the latter equality can be treated invoking with the Parseval equality for the Fourier cosine transform. In fact, calling representation (25) and asymptotic behavior of the modified Bessel function (20), (21) we write

$$\int_0^1 K_{i\tau}(x)\psi(x)\, \frac{dx}{x} = \frac{1}{\cosh\left(\frac{\pi\tau}{2}\right)} \int_0^{\infty} \cos\tau u \int_0^1 \cos(x \sinh u)\psi(x)\, \frac{dxdu}{x}.$$

In the meantime, integrating by parts in the outer integral by u when $|\tau|$ is big and taking into account that the integral by x vanishes when $u \to \infty$ due to the Riemann-Lebesgue lemma,

we obtain

$$\int_0^\infty \cos \tau u \int_0^1 \cos(x \sinh u)\psi(x)\ \frac{dxdu}{x}$$
$$= \frac{1}{\tau}\int_0^\infty \cosh u\ \ \sin \tau u \int_0^1 \sin(x\sinh u)\psi(x)\ dxdu. \tag{44}$$

Moreover, the latter integral converges absolutely and uniformly by $|\tau| > A > 0$ owing to the estimate

$$\int_0^\infty \cosh u\ \left|\sin \tau u \int_0^1 \sin(x \sinh u)\psi(x)\ dx\right| du \le \int_0^\infty \left|\int_0^1 \sin(xy)\psi(x)\ dx\right| dy < \infty$$

and conditions (39) of the theorem. Integrating by parts two more times in the right-hand side of (44) we appeal to the same conditions to derive the asymptotic relation

$$\int\limits_0^1 K_{i\tau}(x)\psi(x)\ \frac{dx}{x} = O\left(\frac{e^{-\frac{\pi}{2}|\tau|}}{\tau^3}\right),\ |\tau| \to \infty.$$

Therefore one can pass to the limit by λ in (43) and then we observe, that $\Psi(t)$ plainly goes to zero when $t \to +\infty$ owing to the Riemann-Lebesgue lemma, completing the proof of the theorem. □

3 An equivalent Salem's problem

In this section we will formulate a problem, which is equivalent to Salem's question [6], having

Corollary 3. *The Fourier-Stieltjes transform*

$$f(t) = \int_0^1 e^{ixt} d?(x)$$

of the Minkowski question mark function vanishes at infinity, i.e. an answer on Salem's question is affirmative, if and only if two limit equalities

$$\lim_{t\to+\infty} t \int_0^\infty ?\left(\frac{1}{x}\right) \sin xt\ dx = 2,$$

$$\lim_{t\to+\infty} t \int_0^\infty ?\left(\frac{1}{x}\right) \cos xt\ dx = 0$$

take place simultaneously.

Proof. It follows immediately from double inequality (17), simple equality due to functional equation (5)

$$\int_0^\infty e^{ixt}\ d?(x) = -\int_0^\infty e^{ixt}\ d?\left(\frac{1}{x}\right)$$

and Corollary 1, where we put $\varphi(x) = ?(1/x),\ x > 0,\ \varphi(0) = 2.$ □

Finally, we generalize Salem's problem, proving

Theorem 3. *Let $k \in \mathbb{N}_0$. If an answer on Salem's question is affirmative, then*

$$f^{(k)}(t) = \int_0^1 (ix)^k e^{itx}\ d?(x) = o(1),\ |t| \to \infty. \tag{45}$$

Proof. It is easily seen that the Fourier- Stieltjes transform of the Minkowski question mark function over $(0,1)$ is infinitely differentiable and so for any $k \in \mathbb{N}_0$ we have (45). Suppose that $f^{(k)}$ does not tend to zero as $|t| \to \infty$. Then we can find a sequence $\{t_m\}_{m=1}^\infty$, $|t_m| \to \infty$ such that

$$\left|\int_0^1 x^k e^{it_m x}\ d?(x)\right| \geq \delta > 0.$$

Let $\frac{t_m}{2\pi} = n_m + \beta_m$, where n_m is an integer and $0 \leq \beta_m < 1$. One can suppose that β_m tends to a limit β, we can always do it choosing again subsequence from $\{t_m\}$ if necessary. Hence

$$|f^{(k)}(t_m)| = \left|\int_0^1 e^{2\pi i\beta x} x^k e^{2\pi i n_m x}\ d?(x)\right| \geq \delta > 0.$$

But this contradicts to Salem's lemma [8], p. 38, because $f(2\pi n) \to 0,\ n \to \infty$ via assumption of the theorem and the Riemann -Stieltjes integral

$$\int_0^1 e^{2\pi i\beta x} x^k\ d?(x)$$

converges for any $k \in \mathbb{N}_0$. □

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